

A note on Riemann normal coordinates.

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Abstract

The goal of this note is to provide a recursive algorithm that allows one to calculate the expansion of the metric tensor up to the desired order in Riemann normal coordinates. We test our expressions up to fourth order and predict results up to sixth order. For an arbitrary number of symmetric partial derivatives acting on the components of the metric tensor subtle treatment is required since the degree of complication increases rapidly.

1 Introduction

Riemann normal coordinates (abbreviated as RNC from now on) have a vast field of applications to Physics. In particular they constitute the main ingredient of the background field method [1] widely used for nonlinear σ - model calculations in curved spacetime. They have the appealing feature that the geodesics passing through the origin have the same form as the equations of straight lines passing through the origin of a Cartesian system of coordinates in Euclidean geometry.

The purpose of this letter is to bridge the gap one confronts when attempts to perform computations that involve expansion of the metric tensor beyond the fourth order [2]. We present the general method for breaking the barrier of the fourth order and at the same time we supply the reader with formulae adequate to reach the order of its choice.

The starting point will be a geodesic, on a compact n -dimensional Riemannian manifold, parametrized by $x^l(s)$ and satisfying the differential equation:

$$\frac{d^2 x^l}{ds^2} + \Gamma_{jk}^l(x) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \quad (1)$$

where s is the arc length and Γ_{jk}^l denotes the Christoffel symbol for the Levi-Civita connection. Any integral curve of (1) is completely determined by a point $\mathcal{P}(x_0^1, x_0^2, \dots, x_0^n)$ and a direction defined by the tangent vector $\xi^l = \left(\frac{dx^l}{ds}\right)_{\mathcal{P}}$. The power series solution of (1) is:

$$x^l(s) = x_0^l + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{d^k x^l}{ds^k} \right)_{\mathcal{P}} s^k \quad (2)$$

and its coefficients can be replaced using successive differentiations by:

$$\left(\frac{dx^l}{ds} \right)_{\mathcal{P}} = \xi^l$$

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$$\begin{aligned}
\left(\frac{d^2 x^l}{ds^2}\right)_P &= -\left(\Gamma_{i_1 i_2}^l\right)_P \xi^{i_1} \xi^{i_2} \\
\left(\frac{d^3 x^l}{ds^3}\right)_P &= -\left[\partial_{(i_1} \Gamma_{i_2 i_3)}^l - 2\Gamma_{(i_1 \rho}^l \Gamma_{i_2 i_3)}^\rho\right]_P \xi^{i_1} \xi^{i_2} \xi^{i_3} \\
\left(\frac{d^4 x^l}{ds^4}\right)_P &= -\left[\partial_{(i_1} \partial_{i_2} \Gamma_{i_3 i_4)}^l - \partial_\rho \Gamma_{(i_1 i_2}^l \Gamma_{i_3 i_4)}^\rho - 4\partial_{(i_1} \Gamma_{i_2 \rho}^l \Gamma_{i_3 i_4)}^\rho - 2\partial_{(i_1} \Gamma_{i_2 i_3}^\rho \Gamma_{\rho i_4)}^l\right. \\
&\quad \left.+ 4\Gamma_{k(i_1}^l \Gamma_{i_2 \rho}^k \Gamma_{i_3 i_4)}^\rho + 2\Gamma_{\rho \sigma}^l \Gamma_{(i_1 i_2}^\rho \Gamma_{i_3 i_4)}^\sigma\right]_P \xi^{i_1} \xi^{i_2} \xi^{i_3} \xi^{i_4}. \\
&\dots
\end{aligned} \tag{3}$$

where dots in (3) represent higher derivative terms at the point \mathcal{P} and symmetrization exclusively relates the lower i indices *. Apparently the domain of convergence of (2) depends on the components g_{ij} and the values of ξ^i . However, for sufficiently small values of s it defines an integral curve of (1).

2 The computational procedure

In RNC the geodesics through \mathcal{P} are straight lines defined by:

$$y^l = \xi^l s. \tag{4}$$

This set of coordinates cannot be used to cover the whole manifold and is valid only for a small neighbourhood of the point \mathcal{P} where the conditions of the existence and uniqueness theorem of differential equation of the geodesics are satisfied. In this region no two geodesics through \mathcal{P} intersect due to one-one correspondence of x^l and y^l . Incidentally, this displays the local nature of RNC.

Substituting (4) into eq. (2) one has:

$$x^l = x_0^l + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\Gamma_{i_1 i_2 \dots i_k}^l\right)_P y^{i_1} y^{i_2} \dots y^{i_k} \tag{5}$$

where $\left(\Gamma_{i_1 i_2 \dots i_k}^l\right)_P$ are the “*generalized Christoffel symbols*” at the point \mathcal{P} used in eq. (3). The Jacobian $\left|\frac{\partial x^l}{\partial y^m}\right|_P \neq 0$ and thus the series (5) can be inverted.

For this local system of coordinates the geodesic equation can be written as:

$$\bar{\Gamma}_{ij}^l(y) y^i y^j = 0 \tag{6}$$

and the power series solution becomes:

$$y^l = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\bar{\Gamma}_{i_1 i_2 \dots i_k}^l\right)_P \xi^{i_1} \xi^{i_2} \dots \xi^{i_k} s^k. \tag{7}$$

Reduction of (7) to (4) for arbitrary values of ξ^l implies that at the origin \mathcal{P} holds:

$$\bar{\Gamma}_{(i_1 i_2 \dots i_k)}^l = 0 \tag{8}$$

*Proof of the fourth expression in (3) would require the use of the identity $\frac{dx^\mu}{ds} D_\mu \frac{dx^\nu}{ds} = 0$

or, by induction, one can easily prove that eq. (8) is equivalent to:

$$\partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{k-2}} \bar{\Gamma}_{i_{k-1} i_k)}^l = 0 \quad (9)$$

Paraphrasing eq. (9) means that all symmetric derivatives of the affine connection vanish at the origin in RNC.

Generally speaking, a covariant second rank tensor field on a manifold can be expanded according to:

$$T_{k_1 k_2}(\tilde{\phi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial}{\partial \xi_{i_1}} \frac{\partial}{\partial \xi_{i_2}} \cdots \frac{\partial}{\partial \xi_{i_n}} \right] T_{k_1 k_2}(\phi) \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n}. \quad (10)$$

The coefficients of the Taylor expansion are tensors and can be expressed in terms of the components R_{mnp}^l [†] of the Riemann curvature tensor and the covariant derivatives $D_k T_{lm}$ and $D_k R_{mnp}^l$. One without much effort can prove that:

$$\begin{aligned} \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-1}} \bar{\Gamma}_{i_n)}^l = & - \left(\frac{n-1}{n+1} \right) \left[D_{(i_1} D_{i_2} \cdots D_{i_{n-2}} \bar{R}_{i_{n-1} k i_n)}^l \right. \\ & + \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-2}} \left(\bar{\Gamma}_{i_{n-1} k}^\alpha \bar{\Gamma}_{\alpha i_n)}^l \right) \\ & - \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-3}} \left(\bar{\Gamma}_{i_{n-2} \alpha}^l \bar{R}_{i_{n-1} k i_n)}^\alpha - l \leftrightarrow \alpha, \alpha \leftrightarrow k \right) \\ & - \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-4}} \left(\bar{\Gamma}_{i_{n-3} \alpha}^l D_{i_{n-2}} \bar{R}_{i_{n-1} k i_n)}^\alpha - l \leftrightarrow \alpha, \alpha \leftrightarrow k \right) \\ & \vdots \\ & \left. - \left(\partial_{(i_1} \bar{\Gamma}_{i_2 \alpha}^l D_{i_3} \cdots D_{i_{n-2}} \bar{R}_{i_{n-1} k i_n)}^\alpha - l \leftrightarrow \alpha, \alpha \leftrightarrow k \right) \right] \end{aligned} \quad (11)$$

where the interchange of covariant and contravariant indices act independently. Expression (11) reproduces for various values of n [‡] the following results:

$$\begin{aligned} \partial_{(i_1} \bar{\Gamma}_{i_2)}^l &= \frac{1}{3} \bar{R}_{(i_1 i_2) k}^l \\ \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3)}^l &= - \frac{1}{2} D_{(i_1} \bar{R}_{i_2 k i_3)}^l \\ \partial_{(i_1} \partial_{i_2} \partial_{i_3} \bar{\Gamma}_{i_4)}^l &= - \frac{3}{5} \left[D_{(i_1} D_{i_2} \bar{R}_{i_3 k i_4)}^l + \frac{2}{9} \bar{R}_{(i_1 i_2 \alpha}^l \bar{R}_{i_3 i_4) k}^\alpha \right] \\ \partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \bar{\Gamma}_{i_5)}^l &= - \frac{2}{3} \left[D_{(i_1} D_{i_2} D_{i_3} \bar{R}_{i_4 k i_5)}^l - D_{(i_1} \bar{R}_{i_2 k i_3}^\alpha \bar{R}_{i_4 i_5) \alpha}^l \right] \\ \partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5} \bar{\Gamma}_{i_6)}^l &= - \frac{5}{7} \left[D_{(i_1} \cdots D_{i_4} \bar{R}_{i_5 k i_6)}^l \right. \\ &\quad - \frac{1}{5} \left(7 D_{(i_1} D_{i_2} \bar{R}_{i_3 \alpha i_4}^l \bar{R}_{i_5 i_6) k}^\alpha + D_{(i_1} D_{i_2} \bar{R}_{i_3 k i_4}^\alpha \bar{R}_{i_5 i_6) \alpha}^l \right) \\ &\quad \left. + \frac{3}{2} D_{(i_1} \bar{R}_{i_2 k i_3}^\alpha D_{i_4} \bar{R}_{i_5 \alpha i_6)}^l - \frac{16}{45} \bar{R}_{(i_1 i_2 \alpha}^l \bar{R}_{i_3 i_4 \beta}^\alpha \bar{R}_{i_5 i_6) k}^\beta \right] \\ &\quad \dots \end{aligned} \quad (12)$$

The coefficients of (10) can be rewritten as:

$$\partial_{(i_1} \partial_{i_2} \cdots \partial_{i_n)} \bar{T}_{k_1 k_2} = D_{(i_1} D_{i_2} \cdots D_{i_n)} \bar{T}_{k_1 k_2}$$

[†]Our conventions are: $R_{mnp}^l = \partial_n \Gamma_{mp}^l + \Gamma_{mp}^k \Gamma_{kn}^l - (n \leftrightarrow p)$, $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ and $R = R^\mu_{\mu}$.

[‡]In [1] there is a misprint for the $n = 4$ case. A minus sign is needed in front of the $\frac{2}{9}$ -term.

$$\begin{aligned}
& + \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-1})} \left[\bar{\Gamma}_{i_n k_1}^\alpha \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right] \\
& + \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-2})} \left[\bar{\Gamma}_{i_{n-1} k_1}^\alpha D_{i_n} \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right] \\
& \vdots \\
& + \partial_{(i_1} \partial_{i_2} \left[\bar{\Gamma}_{i_3 k_1}^\alpha D_{i_4} \cdots D_{i_n} \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right] \\
& + \partial_{(i_1} \left[\bar{\Gamma}_{i_2 k_1}^\alpha D_{i_3} \cdots D_{i_n} \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right]. \tag{13}
\end{aligned}$$

Expressions (11) and (13) compose the building blocks of the current recursive method which produces the following results for different values of n:

$$\begin{aligned}
\partial_{(i_1} \partial_{i_2}) \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2}) \bar{T}_{k_1 k_2} - \frac{1}{3} \left[\bar{R}_{(i_1 k_1 i_2)}^\rho \bar{T}_{\rho k_2} + k_1 \leftrightarrow k_2 \right] \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3}) \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2} D_{i_3}) \bar{T}_{k_1 k_2} \\
&+ \left[\partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1}^\rho \bar{T}_{\rho k_2} + 2 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\rho D_{i_3}) \bar{T}_{\rho k_2} + k_1 \leftrightarrow k_2 \right] \\
&- \frac{1}{3} \left(\bar{R}_{(i_1 k_1 i_2)}^\rho D_{i_3} \bar{T}_{\rho k_2} + k_1 \leftrightarrow k_2 \right) \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4}) \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2} D_{i_3} D_{i_4}) \bar{T}_{k_1 k_2} \\
&+ \left[\partial_{(i_1} \partial_{i_2} \partial_{i_3} \bar{\Gamma}_{i_4 k_1}^\rho \bar{T}_{\rho k_2} + 3 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1}^\rho D_{i_4}) \bar{T}_{\rho k_2} \right. \\
&+ \left. 3 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\rho \left(D_{i_3} D_{i_4} \bar{T}_{\rho k_2} - \frac{1}{3} \left(\bar{R}_{i_3 \rho i_4}^\sigma \bar{T}_{\sigma k_2} + \rho \leftrightarrow k_2 \right) \right) + k_1 \leftrightarrow k_2 \right] \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5}) \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2} D_{i_3} D_{i_4} D_{i_5}) \bar{T}_{k_1 k_2} \\
&- \left[\frac{10}{3} \bar{R}_{(i_1 k_1 i_2)}^\alpha D_{i_3} \cdots D_{i_5} \bar{T}_{\alpha k_2} + 5 D_{(i_1} \bar{R}_{i_2 k_1 i_3}^\alpha D_{i_4} D_{i_5}) \bar{T}_{\alpha k_2} \right. \\
&+ \left. 3 D_{(i_1} D_{i_2} \bar{R}_{i_3 k_1 i_4}^\alpha D_{i_5}) \bar{T}_{\alpha k_2} + \frac{2}{3} D_{(i_1} D_{i_2} D_{i_3} \bar{R}_{i_4 k_1 i_5}^\alpha \bar{T}_{\alpha k_2} \right. \\
&- \frac{2}{3} D_{(i_1} \bar{R}_{i_2 k_1 i_3}^\rho \bar{R}_{i_4 i_5 \rho}^\alpha \bar{T}_{\alpha k_2} + D_{(i_1} \bar{R}_{i_2 k_1 i_3}^\alpha \left(\bar{R}_{i_4 \alpha i_5}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ \left. \frac{2}{3} \left(D_{(i_1} \bar{R}_{i_2 \alpha i_3}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \bar{R}_{i_4 i_5 k_1}^\alpha \pm k_1 \leftrightarrow k_2 \right] \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5} \partial_{i_6}) \bar{T}_{k_1 k_2} &= D_{(i_1} \cdots D_{i_6}) \bar{T}_{k_1 k_2} \\
&+ \left[\partial_{(i_1} \cdots \partial_{i_5} \bar{\Gamma}_{i_6 k_1}^\alpha \bar{T}_{\alpha k_2} + 6 \partial_{(i_1} \cdots \partial_{i_4} \bar{\Gamma}_{i_5 k_1}^\alpha D_{i_6}) \bar{T}_{\alpha k_2} \right. \\
&+ \left. 15 \partial_{(i_1} \cdots \partial_{i_3} \bar{\Gamma}_{i_4 k_1}^\alpha D_{i_5} D_{i_6}) \bar{T}_{\alpha k_2} + 20 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1}^\alpha D_{i_4} \cdots D_{i_6}) \bar{T}_{\alpha k_2} \right. \\
&+ \left. 14 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\alpha D_{i_3} \cdots D_{i_6}) \bar{T}_{\alpha k_2} \right. \\
&+ \left. 10 \partial_{(i_1} \cdots \partial_{i_3} \bar{\Gamma}_{i_4 k_1}^\alpha \left(\partial_{i_5} \bar{\Gamma}_{i_6 \alpha}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \right. \\
&+ \left. 10 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1}^\alpha \left(\partial_{i_4} \partial_{i_5} \bar{\Gamma}_{i_6 \alpha}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \right. \\
&+ \left. 36 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1}^\alpha \left(\partial_{i_4} \bar{\Gamma}_{i_5 \alpha}^\rho D_{i_6}) \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \right. \\
&+ \left. 5 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\alpha \left(\partial_{i_3} \cdots \partial_{i_5} \bar{\Gamma}_{i_6 \alpha}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \right. \\
&+ \left. 24 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\alpha \left(\partial_{i_3} \partial_{i_4} \bar{\Gamma}_{i_5 \alpha}^\rho D_{i_6}) \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \right. \\
&+ \left. 45 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\alpha \left(\partial_{i_3} \bar{\Gamma}_{i_4 \alpha}^\rho D_{i_5} D_{i_6}) \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \right. \\
&+ \left. 15 \left[\partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\alpha \partial_{i_3} \bar{\Gamma}_{i_4 \alpha}^\rho \left(\partial_{i_5} \bar{\Gamma}_{i_6 \rho}^\sigma \bar{T}_{\sigma k_2} + \rho \leftrightarrow k_2 \right) + \alpha \leftrightarrow k_2 \right] \right. \\
&+ \left. \partial_{(i_1} \bar{\Gamma}_{i_2 k_1}^\alpha D_{i_3} \cdots D_{i_6}) \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right]. \tag{14}
\end{aligned}$$

If the second rank tensor with components $\bar{T}_{k_1 k_2}$ is replaced by the metric components $\bar{g}_{k_1 k_2}$ then the related covariant derivatives (provided we deal with a torsion free affine connection) vanish and the above expressions are simplified. One could derive for $n = 5$ the result:

$$\partial_{(i_1} \cdots \partial_{i_5)} \bar{g}_{k_1 k_2} = \frac{4}{3} \left[D_{i_1} \cdots D_{i_3} \bar{R}_{k_1 i_4 i_5 k_2} + 2 \left(D_{i_1} \bar{R}_{k_1 i_2 i_3 \rho} \bar{R}_{i_4 i_5 k_2}^\rho + k_1 \leftrightarrow k_2 \right) \right]. \quad (15)$$

On the other hand for $n = 6$ one gets:

$$\begin{aligned} \partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5} \partial_{i_6)} \bar{g}_{k_1 k_2} = & \frac{10}{7} D_{(i_1} \cdots D_{i_4} \bar{R}_{k_1 i_5 i_6) k_2} + \frac{34}{7} \left(D_{(i_1} D_{i_2} \bar{R}_{k_1 i_3 i_4 \rho} \bar{R}_{i_5 i_6) k_2}^\rho + k_1 \leftrightarrow k_2 \right) \\ & + \frac{55}{7} D_{(i_1} \bar{R}_{k_1 i_2 i_3 \rho} D_{i_4} \bar{R}_{i_5 i_6) k_2}^\rho + \frac{16}{7} \bar{R}_{k_1 (i_1 i_2 \rho} \bar{R}_{i_3 i_4 l}^\rho \bar{R}_{i_5 i_6) k_2}^l. \end{aligned} \quad (16)$$

Thus, plugging into (10) expressions (15) and (16) we end up with the following expansion of the metric tensor in RNC:

$$\begin{aligned} g_{k_1 k_2} = & \bar{g}_{k_1 k_2} + \frac{1}{2!} \frac{2}{3} \bar{R}_{k_1 i_1 i_2 k_2} \xi^{i_1} \xi^{i_2} \\ & + \frac{1}{3!} D_{i_1} \bar{R}_{k_1 i_2 i_3 k_2} \xi^{i_1} \cdots \xi^{i_3} \\ & + \frac{1}{4!} \frac{6}{5} \left[D_{i_1} D_{i_2} \bar{R}_{k_1 i_3 i_4 k_2} + \frac{8}{9} \bar{R}_{k_1 i_1 i_2 m} \bar{R}_{i_3 i_4 k_2}^m \right] \xi^{i_1} \cdots \xi^{i_4} \\ & + \frac{1}{5!} \frac{4}{3} \left[D_{i_1} \cdots D_{i_3} \bar{R}_{k_1 i_4 i_5 k_2} + 2 \left(D_{i_1} \bar{R}_{k_1 i_2 i_3 \rho} \bar{R}_{i_4 i_5 k_2}^\rho + k_1 \leftrightarrow k_2 \right) \right] \xi^{i_1} \cdots \xi^{i_5} \\ & + \frac{1}{6!} \frac{10}{7} \left[D_{i_1} \cdots D_{i_4} \bar{R}_{k_1 i_5 i_6 k_2} + \frac{17}{5} \left(D_{i_1} D_{i_2} \bar{R}_{k_1 i_3 i_4 \rho} \bar{R}_{i_5 i_6 k_2}^\rho + k_1 \leftrightarrow k_2 \right) \right. \\ & + \frac{11}{2} D_{i_1} \bar{R}_{k_1 i_2 i_3 \rho} D_{i_4} \bar{R}_{i_5 i_6 k_2}^\rho + \frac{8}{5} \bar{R}_{k_1 i_1 i_2 \rho} \bar{R}_{i_3 i_4 l}^\rho \bar{R}_{i_5 i_6 k_2}^l \left. \right] \xi^{i_1} \cdots \xi^{i_6} \\ & + O(\xi^{i_1} \cdots \xi^{i_7}). \end{aligned} \quad (17)$$

The expansion (17) is in perfect agreement with that quoted in [3]. The inverse of the metric tensor (obeying $g_{k_1 k_2} g^{k_2 k_3} = \delta_{k_1}^{k_3}$) can be found to be:

$$\begin{aligned} g^{k_1 k_2} = & \bar{g}^{k_1 k_2} - \frac{1}{2!} \frac{2}{3} \bar{R}^{k_1}_{i_1 i_2}{}^{k_2} \xi^{i_1} \xi^{i_2} \\ & - \frac{1}{3!} D_{i_1} \bar{R}^{k_1}_{i_2 i_3}{}^{k_2} \xi^{i_1} \cdots \xi^{i_3} \\ & - \frac{1}{4!} \frac{6}{5} \left[D_{i_1} D_{i_2} \bar{R}^{k_1}_{i_3 i_4}{}^{k_2} - \frac{4}{3} \bar{R}^{k_1}_{i_1 i_2 m} \bar{R}^m_{i_3 i_4}{}^{k_2} \right] \xi^{i_1} \cdots \xi^{i_4} \\ & - \frac{1}{5!} \frac{4}{3} \left[D_{i_1} \cdots D_{i_3} \bar{R}^{k_1}_{i_4 i_5}{}^{k_2} - 3 \left(D_{i_1} \bar{R}^{k_1}_{i_2 i_3 \rho} \bar{R}^\rho_{i_4 i_5}{}^{k_2} + k_1 \leftrightarrow k_2 \right) \right] \xi^{i_1} \cdots \xi^{i_5} \\ & - \frac{1}{6!} \frac{10}{7} \left[D_{i_1} \cdots D_{i_4} \bar{R}^{k_1}_{i_5 i_6}{}^{k_2} - 5 \left(D_{i_1} D_{i_2} \bar{R}^{k_1}_{i_3 i_4 \rho} \bar{R}^\rho_{i_5 i_6}{}^{k_2} + k_1 \leftrightarrow k_2 \right) \right. \\ & - \frac{17}{2} D_{i_1} \bar{R}^{k_1}_{i_2 i_3 \rho} D_{i_4} \bar{R}^\rho_{i_5 i_6}{}^{k_2} + \frac{16}{3} \bar{R}^{k_1}_{i_1 i_2 \rho} \bar{R}^\rho_{i_3 i_4 l} \bar{R}^l_{i_5 i_6}{}^{k_2} \left. \right] \xi^{i_1} \cdots \xi^{i_6} \\ & + O(\xi^{i_1} \cdots \xi^{i_7}). \end{aligned} \quad (18)$$

As a simple check one could consider the symmetric space V_n in RNC for which $D_k \bar{R}_{lmnp} = 0$ and prove that indeed the R terms in (17) satisfy:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^h \frac{2^{2k+2}}{(2k+2)!} f_{\alpha}^{\sigma_1} f_{\sigma_2}^{\sigma_1} \cdots f_{\sigma_{k-1}\beta}^{\sigma_1}, \quad h = \begin{cases} k+1 & \text{if } k \text{ is even} \\ k & \text{if } k \text{ is odd} \end{cases} \quad (19)$$

where $f_{\alpha}^{\sigma_1} = \bar{R}_{\alpha i_1 i_2}^{\sigma_1} \xi^{i_1} \xi^{i_2}$ and $f_{\sigma_{k-1}\beta}^{\sigma_1} = \bar{R}_{i_p \sigma_{k-1} i_{p+1} \beta}^{\sigma_1} \xi^{i_p} \xi^{i_{p+1}}$.

3 Conclusions

We have shown the method one could rely on to evaluate the expansion of the components of the metric tensor in RNC at a specific order. The recursive structure permits an answer the derivation of which becomes cumbersome when one attempts to calculate higher order terms.

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